

# LARGE 2-COLOURED MATCHINGS IN 3-COLOURED COMPLETE HYPERGRAPHS

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ABSTRACT. We prove the generalized Ramsey-type result on large 2-coloured matchings in a 3-coloured complete 3-uniform hypergraph, supporting a conjecture by A. Gyárfás.

## 1. INTRODUCTION AND STATEMENT OF RESULT

In [3], the authors consider generalisations of Ramsey-type problems where the goal is not to find a monochromatic subgraph, but a subgraph that uses “few” colours. In particular, the following theorem is proven:

**Theorem 1** ([3, Theorem 13]). *For  $k \geq 1$ , in every 3-colouring of a complete graph with  $f(k) = \lfloor \frac{7k-1}{3} \rfloor$  vertices there is a 2-coloured matching of size  $k$ . This is sharp for every  $k \geq 2$ , i.e. there is a 3-colouring of  $K_{f(k)-1}$  that does not contain a 2-coloured matching of size  $k$ .*

The example that shows the sharpness of the estimate is close to the colouring obtained by first colouring the vertices with the available colours in proportion close to  $1 : 2 : 4$  and then colouring the edges by the lowest index colour among its endpoints. The analogous question and construction make sense in the case of complete hypergraphs instead of  $K_n$ . At the 1. Emléktábla workshop held at Gyöngyöstarján in July 2010, the first nontrivial case of this question (with 3-uniform hypergraphs and 3 colours) was considered. The best known construction in this case is based on the proportion  $1 : 3 : 9$ , and leads to the following conjecture by A. Gyárfás:

**Conjecture 1.1.** *For any  $t$ -colouring of the complete  $r$ -uniform hypergraph on*

$$n \geq kr + \left\lfloor \frac{(k-1)(t-s)}{1+r+\dots+r^{s-1}} \right\rfloor$$

*vertices, there exists a  $s$ -coloured matching of size  $k$ .*

While it is known that the conjecture fails for e.g.  $t = 6$  and  $s = 2$ , several particular cases are open. We consider here only  $t = 3$ ,  $r = 3$  and  $s = 2$ , in which case the conjecture has the form

**Theorem 2.** *For any 3-colouring of the complete 3-uniform hypergraph on*

$$n \geq 3k + \left\lfloor \frac{k-1}{4} \right\rfloor$$

*vertices, there exists a 2-coloured matching of size  $k$ .*

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The case  $k = 4$  (the first case that is not a trivial consequence of the existing results for the monochromatic problem, see e.g. [1]) was confirmed at the workshop by a team consisting of N. Bushaw, A. Gyárfas, D. Gerbner, L. Merchant, D. Piguet, A. Riet, D. Vu and the author:

**Theorem 3** ([2]). *For any 3-colouring of the complete 3-uniform hypergraph on 12 vertices there exists a perfect matching that uses at most 2 colours.*

In this paper, we prove Theorem 2 in the following equivalent form:

**Theorem 4.** *For any 3-colouring of the complete 3-uniform hypergraph on  $n$  vertices, there exists a 2-coloured matching of size*

$$(1) \quad m(n) = \left\lfloor \frac{4(n+1)}{13} \right\rfloor.$$

It is easy to check that these indeed are formulations of the same result as

$$n = 3k + \left\lfloor \frac{k-1}{4} \right\rfloor = \left\lfloor \frac{13k-1}{4} \right\rfloor$$

is the smallest integer for which  $\left\lfloor \frac{4(n+1)}{13} \right\rfloor \geq k$  holds.

## 2. PROOF

In the proof, the set of vertices of the hypergraph will be denoted by  $V$ , the colouring will be a function  $c : \binom{V}{3} \rightarrow \{1, 2, 3\}$ , and  $\alpha, \beta, \gamma$  will be an arbitrary permutation of the colours 1, 2, 3. The colours are shifted cyclically, e.g. if  $\alpha = 3$ , then  $\alpha + 1$  denotes the colour 1 and if  $\alpha = 1$ , then  $\alpha 1$  denotes the colour 3. A matching on  $n$  vertices is *near perfect*, if it has size  $\lfloor n/3 \rfloor$ .

We call a sextuple  $A$  of points  $\alpha$ -dominated for a colour  $\alpha = 1, 2, 3$ , if for all splittings  $A = B_1 \cup B_2$  into two disjoint triples at least one of  $c(B_1) = \alpha$  or  $c(B_2) = \alpha$  holds. If  $A$  is not  $\alpha$ -dominated for any  $\alpha$ , we call it *universal*. Similarly, we call a set  $X$  of 13 points universal if it admits a near perfect matching in any pair of colours.

The proof proceeds by taking a maximal set of disjoint universal sets  $A_1, \dots, A_l, X_1, \dots, X_m$  with  $|A_i| = 6$  and  $|X_j| = 13$ . If we can now construct a 2-colour matching on  $W = V \setminus (A_1 \cup \dots \cup X_m)$  of the size  $m(|W|)$ , then we can extend it by the appropriately coloured near perfect matchings in the universal sets  $A_i$  and  $X_j$  and keep the size of the matching at least  $m(|V|)$ . Indeed, in the case of an  $A_i$  decreasing  $n$  by 6 decreases  $m(n)$  by at most  $\lceil 4 \cdot 6/13 \rceil = 2$  and in the case of an  $X_j$  decreasing  $n$  by 13 decreases  $m(n)$  by 4. Thus by switching to  $W$  we may assume that there are no universal sets of size 6 or 13, and the resulting structural properties of the colouring will give us the necessary large 2-colour matching.

If a vertex sextuple is  $\alpha$ -dominated, and there are splittings of it into hyperedges of colours  $\alpha$  and  $\alpha + 1$  as well as into those of colours  $\alpha$  and  $\alpha + 2$ , we call this sextuple a *spread* in colour  $\alpha$ , and the splittings are its *demonstration splittings*. Depending of whether the hyperedges of colour  $\alpha$  in the demonstration splittings overlap in 1 or 2 vertices, we assign the spread (with a fixed demonstration splitting implied) a level of 1 or 2 respectively.

**Lemma 4.1.** *Assume that there are two disjoint spreads  $A$  in colour  $\alpha$  and  $B$  in colour  $\beta$  such that  $\alpha \neq \beta$ , and let  $v$  be an arbitrary vertex in the complement*

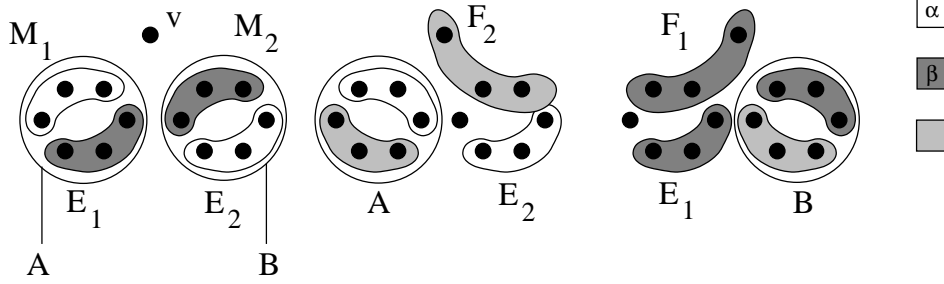


FIGURE 1. If the substitution of a single vertex  $v$  can change the colour of a dominant triple in two spreads of different colour, we have a universal set of size 13. Example: case when  $c(E_1) = \beta$  and  $c(E_2) = \alpha$ .

$V \setminus (A \cup B)$  of their union. Then the following property holds for  $X = A$  or for  $X = B$  (or both): if we substitute  $v$  for any vertex of the dominantly coloured triple in either demonstration splitting of the spread  $X$ , the colour of that triple stays the same, the dominating colour of  $X$ .

*Proof.* Indirectly assume that  $A = M_1 \cup E_1$  and  $B = M_2 \cup E_2$  are splittings which  $v$  “spoils”. That is,  $M_1$  has the dominant colour  $\alpha$  of  $A$  and  $M_1 \cup \{v\}$  contains a triple  $F_1$  of colour different from  $c(M_1) = \alpha$ , and analogously  $c(M_2) = \beta$  and  $M_2 \cup \{v\}$  contains a triple  $F_2$  of colour different from  $\beta$  (see example on Figure 1). Then  $A \cup B \cup \{v\}$  is a universal set of size 13. Indeed, it has a matching of size 4 that contains only colours  $\alpha$  and  $\beta$  since both  $A$  and  $B$  admit perfect matchings in these colours. It also has a matching of size 4 that avoids the colour  $\alpha$ : the spread  $B$  has such a matching of size 2, the triple  $E_1$  has a colour different from  $\alpha$ , and the remainder  $M_1 \cup \{v\}$  contains  $F_1$ , also a triple of a colour different from  $\alpha$ . The same argument with  $A$  and  $B$  reversed produces a near perfect matching that avoids the colour  $\beta$ , proving our claim and arriving at the contradiction that proves the lemma.  $\square$

This coupling property implies a very rigid structure of the colouring:

**Proposition 4.1.** *If there is a pair of disjoint spreads in two different colours, then there is a nearly perfect matching avoiding one colour.*

*Proof.* Without loss of generality we may assume that the two colours are 1 and 2. Let the two spreads be  $A^{(1)}$  (colour 1) and  $A^{(2)}$  (colour 2), and out of all disjoint pairs of spreads of colours 1 and 2 this one contains the most level 2 spreads. Then each of them is either level 2 or it is level 1 and there are no level 2 spreads of their colour that would be disjoint from the other spread.

In both  $A^{(1)}$  and  $A^{(2)}$ , fix two demonstration splittings

$$A^{(i)} = M_+^{(i)} \cup P^{(i)} = M_-^{(i)} \cup N^{(i)}$$

such that  $c(M_+^{(i)}) = c(M_-^{(i)}) = i$ ,  $c(P^{(i)}) = i + 1$  and  $c(N^{(i)}) = i - 1$ . Depending on the level of  $A^{(i)}$ , we can label the vertices of  $A^{(i)} = \{a_1^{(i)}, \dots, a_6^{(i)}\}$  to satisfy the following equalities:

- in case of level 1:

$$\begin{aligned} M_+^{(i)} &= \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}\} & P^{(i)} &= \{v_4^{(i)}, v_5^{(i)}, v_6^{(i)}\} \\ M_-^{(i)} &= \{v_1^{(i)}, v_4^{(i)}, v_5^{(i)}\} & N^{(i)} &= \{v_2^{(i)}, v_3^{(i)}, v_6^{(i)}\} \end{aligned}$$

We will call the vertex  $v_1^{(i)}$  the *dominating* vertex and the rest of the vertices the *core* vertices.

- in case of level 2:

$$\begin{aligned} M_+^{(i)} &= \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}\} & P^{(i)} &= \{v_4^{(i)}, v_5^{(i)}, v_6^{(i)}\} \\ M_-^{(i)} &= \{v_1^{(i)}, v_2^{(i)}, v_4^{(i)}\} & N^{(i)} &= \{v_3^{(i)}, v_5^{(i)}, v_6^{(i)}\} \end{aligned}$$

We will call the vertices  $v_1^{(i)}$  and  $v_2^{(i)}$  the *dominating* vertices and the rest of the vertices the *core* vertices.

In both cases,  $D^{(i)}$  will denote the set of the dominating vertices and  $C^{(i)}$  will denote the set of the core vertices. A pair of vertices will be called *critical*, if they are contained in either  $M_+^{(i)}$  or  $M_-^{(i)}$ .

We choose sets  $\hat{A}^{(1)}$  and  $\hat{A}^{(2)}$  to be a maximal disjoint pair of sets satisfying the following properties:

- $D^{(i)} \subseteq \hat{A}^{(i)} \subseteq V \setminus (C^{(i)} \cup A^{(3-i)})$  for  $i = 1, 2$ .
- For any subset  $D$  of  $\hat{A}^{(i)}$  of size  $|D| = |D^{(i)}|$ , the triples of the sextuple  $C^{(i)} \cup D = (A^{(i)} \setminus D^{(i)}) \cup D$  complementary to  $P^{(i)}$  and  $N^{(i)}$  have colour  $i$ .
- For any subset  $D$  of  $\hat{A}^{(i)}$  of size  $|D| = |D^{(i)}|$ , any pair of vertices  $(u, v) \in V$  that is covered by the complement of either  $P^{(i)}$  or  $N^{(i)}$  in the sextuple  $C^{(i)} \cup D = (A^{(i)} \setminus D^{(i)}) \cup D$ , and any vertex  $w \in \hat{A}^{(i)} \setminus D$ , the triple  $\{u, v, w\}$  has colour  $i$ .

That is,  $\hat{A}^{(i)}$  is a maximal set of vertices (outside of  $\hat{A}^{(3-i)}$ ) extending the set of dominant vertices of  $A^{(i)}$  such that we can switch the dominant vertices of  $A^{(i)}$  with any two vertices of  $\hat{A}^{(i)}$  and still be unable to change the colour of the dominant triples of the modified sextuple by a single vertex change within the set  $\hat{A}^{(i)}$ . Such sets exist (for example,  $D^{(i)}$  satisfies the requirements for  $\hat{A}^{(i)}$ ), and their total size is bounded by  $|V|$ , so we can choose a maximal pair.

We claim that the sets  $\hat{A}^{(1)} \cup \hat{A}^{(2)}$  cover  $V \setminus (C^{(1)} \cup C^{(2)})$ . Indeed, assume that there is a vertex  $w \in V \setminus (C^{(1)} \cup C^{(2)} \cup \hat{A}^{(1)} \cup \hat{A}^{(2)})$  such that it cannot be added to either  $\hat{A}^{(1)}$  or  $\hat{A}^{(2)}$  without violating their defining properties. This means that for  $i = 1, 2$  we can switch the vertices in  $D^{(i)}$  to some other vertices in  $\hat{A}^{(i)}$  in such a way that for the resulting spread  $\tilde{A}^{(i)}$  there is a pair of vertices  $(u^{(i)}, v^{(i)})$  of a dominating triple such that

$$c(u^{(i)}, v^{(i)}, w) \neq i.$$

This contradicts Lemma 4.1 for the spreads  $\tilde{A}^{(1)}$  and  $\tilde{A}^{(2)}$  and the vertex  $w$ .

Additionally, these sets are already easy to colour with 2 colours:

**Lemma 4.2.** *The vertex set  $\hat{A}^{(i)}$  is a clique of colour  $i$ .*

*Proof.* We suppress for brevity the indices  $^{(i)}$ . If  $A$  is level 2, then for any  $\{x, y, z\} \subseteq \hat{A}$  we have by definition of  $\hat{A}$  the property that  $z$  forms triples of colour  $i$  with all the critical vertex pairs of  $C \cup \{x, y\}$ , in particular, with  $\{x, y\}$ , and we are done.

If  $A$  is level 1, recall first that we also assume that there are no spreads of colour  $i$  and level 2 in  $\hat{A} \cup C$ . Indirectly assume furthermore that there is a triple  $X = \{x_1, x_2, x_3\} \subseteq \hat{A}$  such that its colour is not  $i$ . For symmetry reasons it is enough to check the case when  $c(X) = i + 1$ . Then  $P \cup X$  is covered by two disjoint triples of colour  $i + 1$  and must be therefore  $i + 1$ -dominated - otherwise it would form a universal sextuple contrary to our assumptions. But  $P \setminus N = \{v_4, v_5\}$  is a critical pair of vertices and hence  $\{x_1, v_4, v_5\}$  has colour  $i$ ; therefore its complement  $Y = \{x_2, x_3, v_6\}$  has colour  $i + 1$ . This implies that the sextuple  $X \cup N = Y \cup \{x_1, v_2, v_3\}$  can be split into colours  $c(X) = i + 1$  and  $c(N) = i - 1$  as well as into colours  $c(Y) = i + 1$  and  $c(\{x_1, v_2, v_3\}) = i$  (the set  $\{v_2, v_3, x_1\}$  is the complement of  $P$  in  $C \cup \{x_1\}$  with  $x_1 \in \hat{A}$ ), so this sextuple is  $i + 1$ -dominated. Now use the fact that  $\{x_2, v_3\}$  is a critical pair of vertices in  $C \cup \{x_2\}$  (it lies in the complement of  $P$ ). This implies that  $c(\{x_2, x_3, v_3\}) = i$ , and consequently its complement in  $X \cup N$  has colour  $i + 1$ :

$$c(\{x_1, v_2, v_6\}) = i + 1.$$

By definition of  $\hat{A}$ , the sextuple  $\{x_1\} \cup C$  is  $i$ -dominant as it cannot be dominant in any other colour. Hence the complement of  $\{x_1, v_2, v_6\}$  in it has to have colour  $i$ :

$$c(\{v_3, v_4, v_5\}) = i.$$

Also,  $\{v_4, v_5\}$  is a critical pair of vertices, so we have

$$c(\{x_1, v_4, v_5\}) = i$$

as well. But this means that  $C \cup \{x_1\}$  is a level 2 spread of colour  $i$  as evidenced by splitting into  $\{x_1, v_4, v_5\} \cup N$  (colours  $i$  and  $i - 1$  respectively) and into  $\{v_3, v_4, v_5\} \cup \{x_1, v_2, v_6\}$  (colours  $i$  and  $i + 1$  respectively) - a contradiction with our initial assumption, hence  $\hat{A}$  is indeed a clique of colour  $i$  as claimed.  $\square$

This also implies that  $\hat{A}^{(1)} \cup M_+^{(1)}$  is a clique of colour 1 and  $\hat{A}^{(2)} \cup M_-^{(2)}$  is a clique of colour 2 (we are adding a vertex or a critical pair of vertices to the appropriate  $\hat{A}^{(i)}$ ). Notice that their complement is the union of the 2-coloured hyperedge  $P^{(1)}$  and the 1-coloured hyperedge  $N^{(2)}$ .

**Lemma 4.3.** *If  $U$  and  $W$  are disjoint cliques of colours 1 and 2 respectively such that  $|U| \geq 3$  and  $|W| \geq 3$ , then there exists an almost perfect matching in  $U \cup W$  in colours 1 and 2.*

*Proof.* If  $|U| + |W| \bmod 3 = |U| \bmod 3 + |W| \bmod 3$ , that is,  $|U| \bmod 3 + |W| \bmod 3 \leq 2$ , then taking maximal disjoint sets of hyperedges in  $U$  and  $W$  separately gives an almost perfect matching in colours 1 and 2.

If this is not the case, then both  $|U| \bmod 3$  and  $|W| \bmod 3$  are at least 1 and at least one of them is equal to 2; without loss of generality, we may assume that  $|U| \equiv 2 \bmod 3$ . We claim that there is a hyperedge  $E \subset U \cup W$  of colour 1 or 2 with the property that  $|U \cap E| = 2$ . Indeed, assume indirectly that all triples intersecting  $U$  in 2 vertices and  $W$  in 1 vertex have colour 3. Since  $|U| \geq 3$  and  $|U| \bmod 3 = 2$ , we have  $|U| \geq 5$ . Consider any four distinct vertices  $u_1, u_2, u_3, u_4 \in U$  and any two distinct vertices  $w_1, w_2 \in W$ . Then the set  $X = \{u_1, u_2, u_3, u_4, w_1, w_2\}$  is covered by the triples  $\{u_1, u_2, w_1\}$  and  $\{u_3, u_4, w_2\}$ , both of which have to have colour 3. Hence  $X$  can only be 3-dominated, consequently at least one of the members of the matching  $\{u_1, w_1, w_2\} \cup \{u_2, u_3, u_4\}$  has colour 3. But the triple  $\{u_2, u_3, u_4\}$  lies in

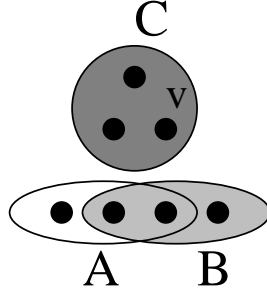


FIGURE 2. If there are no spreads, then only two colours may be used.

the clique  $U$  and therefore has colour 1, so  $c(\{u_1, w_1, w_2\}) = 3$ . This implies that for any choice of a vertex  $w_3 \in W \setminus \{w_1, w_2\}$  we have on one hand

$$c(\{u_1, w_1, w_2\}) = 3 \text{ and } c(\{u_2, u_3, w_3\}) = 3$$

due to the latter triple intersecting  $U$  in 2 vertices, and on the other hand

$$c(\{u_1, u_2, u_3\}) = 1 \text{ and } c(\{w_1, w_2, w_3\}) = 2$$

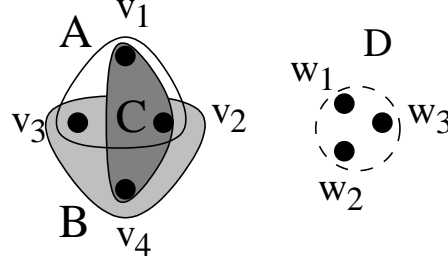
due to  $U$  and  $V$  being cliques. Hence  $\{u_1, u_2, u_3, w_1, w_2, w_3\}$  would be a universal sextuple, a contradiction that proves our claim.

Given a hyperedge  $E \subset U \cup W$  of colour 1 or 2 with the property that  $|U \cap E| = 2$ , we can just add it to the union of any maximal matching of  $U \setminus E$  and any maximal matching of  $W \setminus E$  to get a nearly perfect matching of  $U \cup W$  in colours 1 and 2.  $\square$

Applying Lemma 4.3 to the cliques  $\hat{A}^{(1)} \cup M_+^{(1)}$  and  $\hat{A}^{(2)} \cup M_-^{(2)}$  and adding the triples  $P^{(1)}$  and  $N^{(2)}$  yields a near perfect matching in colours 1 and 2 on  $V$ . This finishes the proof of Proposition 4.1.  $\square$

Once we can exclude two disjoint spreads of different colours, we have two possibilities: either there are no spreads at all, or there is a spread of, say, colour 1, and any spread in its complement is also of colour 1. We will also assume that  $|V| \geq 9$  as otherwise the 2-colour condition is trivially fulfilled by any near perfect matching.

**Case 1: there are no spreads.** If there are no spreads, then no sextuple can contain triples of all three colours: one of them would be dominating, and any two instances of the other two colours could be chosen to be  $P$  and  $N$  of a spread. We will first look for a pair of triples of different colours that share two vertices,  $c(A) \neq c(B)$ ,  $|A \cap B| = 2$ . If there are no such pairs, then all triples have the same colour and any nearly perfect matching is monochromatic, we are done. If, on the other hand, such triples  $A$  and  $B$  exist, we may assume without loss of generality that  $c(A) = 1$  and  $c(B) = 2$ . Could there be triples of colour 3 (see Figure 2)? Any such triple  $C$  would have to be disjoint from  $A \cup B$ , because otherwise their union  $A \cup B \cup C$  (together with any other vertex if it has only 5 elements) would form a sextuple of vertices that contains all the three colours. Then for any vertex  $v \in C$  the triple  $T = (A \cap B) \cup \{v\}$  is covered by both  $A \cup C$  (covering only triples of colour 1 and 3) and  $B \cup C$  (covering only triples of colour 2 and 3) and therefore

FIGURE 3. Case  $|C \cap B| = 2$ .

can only be of colour 3. But then  $A \cup B \cup \{v\}$  together with any other vertex form a sextuple that contains triples of all three colours,  $A$ ,  $B$  and  $T$  - a contradiction.

Therefore in this case only two colours may be used at all, so any near perfect matching automatically satisfies our desired condition.

**Case 2: there exists a spread** (of colour 1, say). We first investigate what happens if there are no spreads of other colour at all. This results in a highly ordered structure:

**Proposition 4.2.** *If a colouring is such that all spreads are of colour 1, then either*

- *there exists a near perfect matching avoiding colour 2 or colour 3, or*
- *there are no triples of colour 1 at all.*

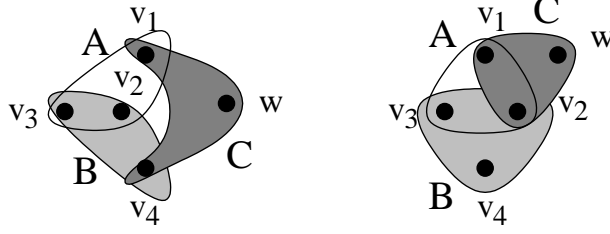
*Proof.* First note that the condition on the spreads means that whenever a sextuple contains triples of all three colours, it is 1-dominated. In particular, if a triple is covered by a disjoint union of a 2-coloured and a 3-coloured triple, it cannot have colour 1 - the union in question can only be 2- or 3-dominated. We show that this statement can also be used for non-disjoint pairs of triples of colours 2 and 3.

**Lemma 4.4.** *Assume that all spreads in the colouring are of colour 1. Then either*

- *there are no triples of colour 1 covered by the union of a triple of colour 2 and a triple of colour 3, or*
- *there exists a near perfect matching avoiding colour 2 as well as one avoiding colour 3.*

*Proof.* Assume  $A = \{v_1, v_2, v_3\}$  is a colour 1 triple that is covered by triples  $B$  and  $C$  of colours 2 and 3 respectively. At least one of these has to cover 2 vertices of  $A$ , so after a renumbering of colours, triples and vertices we may assume that  $B = \{v_2, v_3, v_4\}$  and  $v_1 \in C$ . We now have three cases for the situation of  $C$  with respect to  $A$  and  $B$ :

- (1)  $C \cap B = \emptyset$ . Then  $B$  and  $C$  are disjoint triples of colour 2 and 3 respectively which cover  $A$ , a triple of colour 1 - a contradiction.
- (2)  $C \cap B = \{v_2, v_4\}$  (or analogously  $\{v_3, v_4\}$ ); that is,  $C$  is covered by  $A \cup B$  (see Figure 3). The union  $A \cup B = A \cup B \cup C$  has 4 elements and contains all three colours, so adding any pair of vertices  $x, y$  makes it a 1-dominated sextuple. In this sextuple, the triples  $\{x, y, v_1\}$  and  $\{x, y, v_3\}$  have non-1-coloured complements, so they have to have colour 1 themselves. Now assume there is a triple  $D = \{w_1, w_2, w_3\}$  of colour 2 disjoint from  $A \cup B$  (the case of  $c(D) = 3$  is similar). Then  $D \cup C$  is a disjoint union of a

FIGURE 4. Case  $|C \cap B| = 1$ .

2-coloured triple and a 3-coloured one, and it covers the 1-coloured triple  $\{w_1, w_2, v_1\}$  - a contradiction. Hence all triples disjoint from  $A \cup B$  have colour 1. Consequently we can choose a near perfect matching either in colour 1 only, or at will in colours 1 and 2, or in colours 1 and 3 - if the total number of vertices is congruent to 1 or 2 modulo 3, we take a near perfect matching in the complement of  $A \cup B$  and add  $A$ , otherwise we take a near perfect matching in the complement of  $A \cup B$ , add the triple  $B$  or  $C$  depending on which colour out of 2 and 3 is wanted and match up the remaining two vertices with either  $v_1$  or  $v_3$  (whichever is left out).

- (3)  $|C \cap B| = 1$ ; let  $w$  denote the single vertex in  $C \setminus (A \cup B)$  (see Figure 4). By the same argument as before, for any vertex  $x$  not in  $A \cup B \cup C$  we have that  $A \cup B \cup C \cup \{x\}$  is 1-dominated, so the complements of the non-1-coloured triples  $B$  and  $C$  must have colour 1:

$$c(\{x\} \cup ((A \cup B \cup C) \setminus B)) = 1,$$

$$c(\{x\} \cup ((A \cup B \cup C) \setminus C)) = 1.$$

This makes it impossible to have triples of colour other than 1 disjoint from  $A \cup B \cup C$ , as they would cover a 1-coloured triple together with either  $B$  or  $C$  (whichever has the colour other from that of the selected triple). Now taking a maximal matching outside  $A \cup B \cup C$ , we can extend it to a near perfect matching avoiding the colour 2 or the colour 3 as follows. If there are no vertices left outside the matching, add  $A$  to get a 1-coloured matching. If there is 1 vertex left, join it to  $(A \cup B \cup C) \setminus B$  and add  $B$  to avoid the colour 3; do the same with  $B$  and  $C$  switched to avoid the colour 2. Finally, if there are 2 vertices left, join them respectively to the disjoint vertex pairs  $(A \cup B \cup C) \setminus B$  and  $(A \cup B \cup C) \setminus C$  to obtain a matching of colour 1.

□

Thus we may restrict our attention to the case when the union of a triple of colour 2 and a triple of colour 3 cannot cover a triple of colour 1, even if they are not disjoint.

We now try to find a vertex such that all triples containing it are of colour 1; we will call such a vertex *1-forcing*. If there are no triples of colours 1 and 2 or 1 and 3 such that they intersect in two vertices, then either there are no triples of colour 1 - in which case there is a near perfect matching in colours 2 and 3 - or there are no triples of colour different from 1 - in which case there is a near perfect matching in colour 1. Hence we might assume that there is a pair of triples of the form  $A = \{v_1, v_2, v_3\}$ ,  $B = \{v_2, v_3, v_4\}$  with  $c(A) = 1$  and  $c(B) = 2$ , say. By our



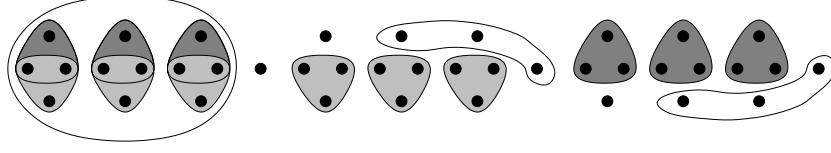


FIGURE 5. A 1-forcing vertex with 3 disjoint “neighbouring” triples of colours 2 and 3 implies the existence of a universal 13-vertex set.

previous lemma there are no triples of colour 3 that contain  $v_1$ . If there are no such triples disjoint from  $A \cup B$  either, then any near perfect matching containing  $A$  or  $B$  will be 1 and 2-coloured. Assume therefore that there is a triple  $C$  of colour  $c(C) = 3$  in the complement of  $A \cup B$ . No triple covered by  $B \cup C$  can have colour 1, in particular the triple  $D = \{v_2, v_3, x\}$  with some  $x \in C$  has to have colour 2 or 3. If  $c(D) = 3$ , then we can repeat the argument with  $A$  and  $D$  instead of  $A$  and  $B$  to get that no colour 2 triples contain  $v_1$  either, so  $v_1$  is 1-forcing. If  $c(D) = 2$ , then  $A \cup C$  contains triples of all three colours and thus is 1-dominated, in particular the triple  $E = \{v_1\} \cup (C \setminus \{x\}) = (A \cup C) \setminus D$  has to have colour 1 due to its complement having colour 2. In this case, the application of the same argument to  $E$  and  $C$  gives us the same result of no triples of colour 2 containing  $v_1$  and  $v_1$  is 1-forcing again.

Putting such a 1-forcing vertex aside and repeating the procedure, we end up with a set of 1-forcing vertices and a remainder set where either there are no triples of colour 1 or there is a near perfect matching in colours 1 and 2 or 3. In the latter case, we can just complete the matching with 1-forcing vertices at will, so we assume now that the remainder, denoted henceforth by  $R$ , only has triples of colours 2 and 3.

If  $R = V$ , then we get the second conclusion of our proposition, so we may assume that there is at least one 1-forcing vertex. If, moreover,  $R$  had three disjoint pairs of triples of colours 2 and 3 that intersect in 2 vertices, we could add a 1-forcing vertex and get a universal 13-vertex set (see Figure 5) - a contradiction. If there are no three disjoint pairs like that, then after picking at most two of them the rest (denoted by  $R'$ ) has to be a clique, of colour 2, say. We can then take a 1-forcing vertex and add to it those vertices of the triples of colour 3 among the chosen pairs of colour 2 and 3 that are not covered by the corresponding triples of colour 2, and add another vertex from  $R'$  if we still don't have three vertices. Choose a near perfect matching from the rest of  $R$  containing the selected triples of colour 2 and then cover the rest with 1-forcing vertices if there are any left. This yields a near perfect matching in colours 1 and 2, and finishes the proof of the proposition.  $\square$

Since in both cases of Proposition 4.2 we get a near perfect matching in 2 colours, we only need to consider the case where there exist spreads of a different colour. By symmetry, assume that  $U$  is a spread of colour 1 and  $W$  is a spread of colour 2. By Proposition 4.1, we can apply Proposition 4.2 to  $V \setminus U$ , so we either get a near perfect matching avoiding colour 2 or 3 or no triples of colour 1 at all. In the first case, we can add one of the demonstration splittings of  $U$  to get a near perfect matching of  $V$  avoiding either colour 2 or colour 3. The same argument of applying Proposition 4.2 to  $V \setminus W$  yields either a near perfect matching on  $V$  in 2 colours or

no triples of colour 2 at all. We may hence assume that we got the second result in both attempts, and the colouring is such that all triples of colour 1 intersect  $U$  while all triples of colour 2 intersect  $W$ .

We see that in such a setup, the vertex set  $V \setminus (U \cup W)$  is a clique in colour 3. Additionally, there is at least one triple of colour 3 in  $U$  (and in  $W$ , but it may not be disjoint from those in  $U$ ), so if we take a maximal matching in colour 3 extending an exhausting of  $V \setminus (U \cup W)$ , we end up with at most  $2 + |U| + |W| - |U \cap W| - 3 \leq 10$  vertices not covered by this matching and consequently only containing triples of colours 1 and 2. We distinguish between three possibilities for the number  $m$  of vertices left out:

- $m \leq 8$  and  $m \neq 6$ . By the theorem of Alon, Frankl and Lovász ([1]) there is an almost perfect monochromatic matching in this 2-coloured subgraph: 3 vertices needed for 1 triple, 7 for two triples. Adding it to the initial colour 3 matching, we obtain a near perfect matching of  $V$  in 2 colours.
- $m = 6$  or  $m = 9$ . In this case  $|V|$  is a multiple of 3, so either it is at most 12, in which case we apply Theorem 3, or  $|V|$  is at least 15, hence the prediction (1) gives a size at least one less than that of a perfect matching. In this latter case, the result of [1] is sufficient (a size 1 matching for  $m = 6$  and a size 2 matching for  $m = 9$ ).
- $m = 10$ . Here all of our estimates have to be sharp, that is,  $|U \cap W| = 1$  and we must have 2 vertices from  $V \setminus (U \cup W)$  and 8 vertices from  $U \cup W$  not covered by the matching in colour 3. If choosing a different maximal matching in colour 3 leads to a different case, we are done, so we may assume that no matter which 2 vertices  $a$  and  $b$  of  $V \setminus (U \cup W)$  are left out from the initial matching, there do not exist 2 disjoint triples of colour 3 in  $U \cup W \cup \{a, b\}$ . But any vertex in  $U \cup W \setminus (U \cap W)$  lies in the complement of a triple of colour 3 - the elements of  $U \setminus W$  miss the colour 3 triple in  $W$  and vice versa. Therefore any vertex in  $U \cup W \setminus (U \cap W)$  together with any two vertices in  $V \setminus (U \cup W)$ , and any two vertices in  $U \setminus W$  (or  $W \setminus U$ ) together with any vertex in  $V \setminus (U \cup W)$ , give a hyperedge of colour 1 or 2.

If now  $|V| \geq 31$ , we can cover all of  $V \setminus (U \cap W)$  (at most 30 vertices) by at most 10 such hyperedges (adding a suitable splitting of  $W$  or applying Theorem 3 if  $|V| = 13$ ). If, on the other hand,  $|V| \geq 32$ , then the formula (1) predicts a matching at least 1 less than a near perfect one. Such a matching can be found with direct application of [1] to the 10-vertex remainder as before.

In all three cases we arrive at a matching in 2 colours of size at least that predicted by (1), finishing the proof of Theorem 4.

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